Linear Algebra I

22/01/2019, Tuesday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Systems of linear equations $(1+4+3+(1+3+3) = 15$ pts)

Consider the following system of linear equations in the unknowns v, w, x, y, and z where α is a real number:

$$\begin{array}{rcrcrcrcrcl} 2v & + & 2w & + & 2x & + & 4y & & & = & 2 \\ & & 2w & + & 4x & + & 2y & + & 2z & = & 0 \\ 4v & + & w & + & & & 5y & + & z & = & 4 \\ 6v & + & 3w & + & 2x & + & 9y & + & z & = & \alpha \end{array}$$

(a) Write down the augmented matrix.

(b) By performing elementary row operations, put the augmented matrix into row echelon form.

(c) Determine all values of α so that the system is consistent.

(d) For the values of α found above,

(i) determine the *lead* and *free* variables.

(ii) put the augmented matrix into *reduced* row echelon form by performing elementary row operations.

(iii) find the solution set.

Required Knowledge: **Gauss-elimination, row operations, row echelon form, consistency, and set of solutions.**

Solution:

1a: The augmented matrix is given by

$$\left[\begin{array}{ccccc|c} 2 & 2 & 2 & 4 & 0 & 2 \\ 0 & 2 & 4 & 2 & 2 & 0 \\ 4 & 1 & 0 & 5 & 1 & 4 \\ 6 & 3 & 2 & 9 & 1 & \alpha \end{array}\right].$$

1b: By applying elementary row operations, we obtain:

$$\left[\begin{array}{ccccc|c} 2 & 2 & 2 & 4 & 0 & 2 \\ 0 & 2 & 4 & 2 & 2 & 0 \\ 4 & 1 & 0 & 5 & 1 & 4 \\ 6 & 3 & 2 & 9 & 1 & \alpha \end{array}\right] \xrightarrow[\textcircled{2}=\frac{1}{2}\cdot\textcircled{2}]{\textcircled{1}=\frac{1}{2}\cdot\textcircled{1}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 5 & 1 & 4 \\ 6 & 3 & 2 & 9 & 1 & \alpha \end{array}\right] \xrightarrow[\textcircled{4}=\textcircled{4}-6\cdot\textcircled{1}]{\textcircled{3}=\textcircled{3}-4\cdot\textcircled{1}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & \alpha-6 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & \alpha-6 \end{array}\right] \xrightarrow{\textcircled{4}=\textcircled{4}-1\cdot\textcircled{3}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha-6 \end{array}\right]$$

$$\left[\begin{array}{rrrrr|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha-6 \end{array}\right] \xrightarrow{\textcircled{3} = \textcircled{3} + 3\cdot\textcircled{1}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha-6 \end{array}\right]$$

$$\left[\begin{array}{rrrrr|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -3 & -4 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha-6 \end{array}\right] \xrightarrow{\textcircled{3} = \frac{1}{2}\cdot\textcircled{3}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha-6 \end{array}\right].$$

This leads to row echelon form of

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

if $\alpha = 6$ and

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

if $\alpha \neq 6$ by diving the last row by $\alpha - 6$.

1c: The system is consistent if and only if $\alpha = 6$ and inconsistent otherwise.

1d(i): The lead variables are v, w, and x whereas y z are free variables.

1d(ii): By applying elementary row operations when $\alpha = 6$, we obtain:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\textcircled{1} = \textcircled{1} - \textcircled{3}]{\textcircled{2} = \textcircled{2} - 2\cdot\textcircled{3}} \left[\begin{array}{rrrrr|c} 1 & 1 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\left[\begin{array}{rrrrr|c} 1 & 1 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\textcircled{1} = \textcircled{1} - \textcircled{2}} \left[\begin{array}{rrrrr|c} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1d(iii): From the row reduced echelon form, we see that the general solution is given by

$$\begin{aligned} v &= 1 - y - z \\ w &= -y + 3z \\ x &= -2z. \end{aligned}$$

2 Determinants (9 + 6 = 15 pts)

Consider the matrix

$$M = \begin{bmatrix} \alpha & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where α is real number.

(a) By using *only* row/column operations, find the determinant of M.

(b) Determine all values of α such that M is nonsingular.

REQUIRED KNOWLEDGE: **Determinants, row/column operations, nonsingularity.**

SOLUTION:

2a: By subtracting the last row from the first three rows, we see that

$$\begin{vmatrix} \alpha & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \alpha - 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

Now, we subtract the third row from the first two and obtain:

$$\begin{vmatrix} \alpha - 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \alpha - 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

Finally, we subtract the second from the first:

$$\begin{vmatrix} \alpha - 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \alpha - 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

Since the last matrix we obtain is a triangular matrix, its determinant is the product of diagonal entries. As such we have:

$$\begin{vmatrix} \alpha & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \alpha - 3.$$

2b: A square matrix is nonsingular if and only is its determinant is nonzero. Therefore, M is nonsingular if and only if $\alpha \neq 3$.

3 Least squares problem (15 pts)

Find the line of the form $y = a + bx$ that gives the best least squares approximation to the points:

x	1	1	0
y	0	1	1

REQUIRED KNOWLEDGE: **Least-squares problem, normal equations.**

SOLUTION:

The corresponding least squares problem is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

This leads to the following normal equations:

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, we obtain the least squares solution as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}.$$

4 Vectors spaces

$((3+4)+(4+4)=15 \text{ pts})$

(a) Let J be the 3×3 matrix given by

$$J=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}.$$

(i) Is the set $S=\{A\in\mathbb{R}^{3\times 3}\mid AJ=JA\}$ a subspace?

(ii) If it is so, find a basis for S and determine its dimension.

(b) Let M be a 2×2 matrix.

(i) Let $L_M:\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}$ given by $L_M(X)=MX+XM$. Show that L_M is a linear transformation.

(ii) Take $M=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. Write down the matrix representation of L_M using the following basis for $\mathbb{R}^{2\times 2}$:

$$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}.$$

Required Knowledge: **Vector spaces, subspaces, basis, dimension, linear transformations, matrix representations.**

Solution:

4a(i): To show that S is a subspace, we begin with the observation that $0_{3\times 3}\in S$, that is $S\neq\varnothing$. Let α be a scalar and $A\in S$. Note that

$$(\alpha A)J=\alpha AJ=\alpha JA=J(\alpha A).$$

Hence, we obtain $\alpha A\in S$. This means that S is closed under scalar multiplication. Now, let A and B belong to S and note that

$$(A+B)J=AJ+BJ=JA+JB=J(A+B).$$

Thus, we see that $A+B\in S$. This means that S is closed under vector addition. Consequently, S is a subspace.

4a(ii): Let

$$A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}.$$

Note that

$$AJ=\begin{bmatrix}0&a&b\\0&d&e\\0&g&h\end{bmatrix}$$

and

$$JA=\begin{bmatrix}d&e&f\\g&h&i\\0&0&0\end{bmatrix}.$$

Therefore, $A\in S$ if and only if $d=g=h=0$ and $a=e=i$ and $b=f$. In other words, $A\in S$ if and only if

$$A=\begin{bmatrix}\alpha&\beta&\gamma\\0&\alpha&\beta\\0&0&\alpha\end{bmatrix}$$

for some α, β, and γ. This means that $A \in S$ if and only if

$$A = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some α, β, and γ. Therefore, S is spanned by

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Since these three matrices are linearly independent in $\mathbb{R}^{3\times 3}$, we can conclude that they form a basis. The dimension of a subspace is the cardinality of a set of basis vectors. Therefore, S is 3 dimensional.

4b(i): Let α and β be scalars and X and Y belong to $\mathbb{R}^{2\times 2}$. Note that

$$L_M(\alpha X+\beta Y) = M(\alpha X+\beta Y)+(\alpha X+\beta Y)M = \alpha(MX+XM)+\beta(MY+YM) = \alpha L_M(X)+\beta L_M(Y).$$

Therefore, L_M is a linear transformation.

In order to find the matrix representation, we need to apply the transformation on the basis vectors:

$$\begin{aligned}
L_M(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2a & b \\ c & 0 \end{bmatrix} \\
&= 2a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
L_M(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} + \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} c & a+d \\ 0 & c \end{bmatrix} \\
&= c \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (a+d) \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
L_M(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} b & 0 \\ a+d & b \end{bmatrix} \\
&= b \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (a+d) \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
L_M(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ c & 2d \end{bmatrix} \\
&= 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2d \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

This leads to the following matrix representation:

$$\begin{bmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{bmatrix}.$$

5 Characteristic polynomial, determinant, and trace (3 + 4 + 4 + 4 = 15 pts)

Let M be the 3×3 matrix given by

$$M = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

where a, b, and c are real numbers.

(a) By using the relationship between the determinant and eigenvalues of a matrix, show that 0 is an eigenvalue of M.

(b) By using the definition of eigenvalue, show that $a + b + c$ is an eigenvalue of M.

(c) By using the relationship between the trace and eigenvalues of a matrix, show that the characteristic polynomial of M is given by $p_M(\lambda) = \lambda^2(\lambda - a - b - c)$.

(d) Suppose that $a + b + c \neq 0$. Show that M is diagonalizable.

REQUIRED KNOWLEDGE: **Characteristic polynomial, eigenvalues, determinant, and trace.**

SOLUTION:

5a: First, we observe that M is singular. Indeed, if one of the real numbers a, b, or c is zero, then M has a zero column and hence $\det(M) = 0$. If none of them is zero, then

$$M \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} = 0$$

and hence $\det(M) = 0$. Since the determinant is equal to the product of eigenvalues, we can then conclude that 0 must be an eigenvalue of M.

5b: Note that

$$M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (a + b + c) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, we see that $a + b + c$ is an eigenvalue of M.

5c: First, we claim that M cannot have two distinct nonzero eigenvalues. To see this, let $\lambda \neq 0$ and note that

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then, we have $ax + by + cz = \lambda x = \lambda y = \lambda z$. This results in $x = y = z$ since $\lambda \neq 0$. Since eigenvectors corresponding to distinct eigenvalues must be linearly independent, we see that M does not have two distinct nonzero eigenvalues. Therefore, there are two possibilities: either all eigenvalues are zero or two of them are zero and one is nonzero. Since the trace of a square matrix equals the sum of the eigenvalues, we have either all eigenvalues are zero and $a + b + c = 0$ or $a+b+c \neq 0$ is the only nonzero eigenvalue. Clearly, we have $p_M(\lambda) = \lambda^2(\lambda-a-b-c)$ in both cases.

5d: From (c), we know that there are two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a + b + c \neq 0$. Also from (c), we know that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to λ_2. Next, we find eigenvectors corresponding to $\lambda_1 = 0$. Note that

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $ax + by + cz = 0$. Since $a + b + c \neq 0$, at one of the numbers a, b, and c must be nonzero. Suppose that $a \neq 0$. Then, we see that

$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we have the following eigenvectors for M:

$$\left\{ \begin{bmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Note that

$$\begin{vmatrix} -\frac{b}{a} & -\frac{c}{a} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} -\frac{b}{a} & 1 \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -\frac{b}{a} & -\frac{c}{a} \\ 1 & 0 \end{vmatrix} = \frac{b}{a} + 1 + \frac{c}{a} = \frac{a + b = c}{a} \neq 0.$$

Therefore, M has 3 linearly independent eigenvectors and hence is diagonalizable.

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6 Eigenvalues/vectors $(4+5+2+4=15 \text{ pts})$

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of M.

(b) Show that M is diagonalizable.

(c) Find a matrix X that diagonalizes M.

(d) Find e^M by using the matrix X.

REQUIRED KNOWLEDGE: **Eigenvalues, eigenvectors, and diagonalization, matrix exponential.**

SOLUTION:

6a: Note that

$$\det(\lambda I - M) = \det(\begin{bmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ -1 & 0 & \lambda-1 \end{bmatrix}) = (\lambda-2)\det(\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{bmatrix}) = (\lambda-2)\big((\lambda-1)^2-1\big) = (\lambda-2)(\lambda-2)\lambda.$$

Therefore, the eigenvalues are given by $\lambda_1 = 0$ and $\lambda_2 = 2$.

6b: First, we need to find eigenvectors.

For $\lambda_1 = 0$, we have

$$(\lambda_1 I - M)\boldsymbol{x} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This leads to $x_1 + x_3 = 0$ and $x_2 = 0$. Hence, an eigenvector is given by

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = 2$, we have

$$(\lambda_2 I - M)\boldsymbol{y} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This leads to $y_1 = y_3$. Therefore, we obtain

$$\boldsymbol{y} = \begin{bmatrix} a \\ b \\ a \end{bmatrix}.$$

Hence, we see that

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent eigenvectors for $\lambda_2 = 2$.

Note that

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

Therefore, M has 3 linearly independent eigenvectors and hence is diagonalizable.

6c: Diagonalizers can be found from eigenvectors. Indeed, if we take

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

we see that

$$MX = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = XD.$$

6d: Since $M = XDX^{-1}$, we know that $e^M = Xe^DX^{-1}$. Therefore, we first need to find the inverse of X:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\textcircled{3} = \textcircled{3} + \textcircled{1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\textcircled{3} = \frac{1}{2} \cdot \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{\textcircled{1} = \textcircled{1} - 1 \cdot \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

Thus, we see that

$$X^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Consequently,

$$\begin{aligned} e^M &= Xe^DX^{-1} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & e^2 \\ 0 & e^2 & 0 \\ -1 & 0 & e^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(e^2+1) & 0 & \frac{1}{2}(e^2-1) \\ 0 & e^2 & 0 \\ \frac{1}{2}(e^2-1) & 0 & \frac{1}{2}(e^2+1) \end{bmatrix}. \end{aligned}$$